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ON A PARTICULAR CRITERION OF OPTIMAL PURSUIT TIME

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Effective criteria are presented for testing the sufficient condition of optimal pursuit time (time of first absorption) for linear differential games that satisfy the conditions of local convexity. The class of problems for which the sufficient condition is also the necessary one is indicated.

1. Let the linear problem of pursuit in an n-dimensional Euclidean space R be defined by

a) the linear vector differential equation

$$dz / dt = Cz - u + v \tag{1.1}$$

where C is a constant square matrix of order n; $u = u(t) \in P$ and $v = v(t) \in Q$ are vector functions measurable for $t \geqslant 0$, which are called the controls of players (the pursuer and the pursued, respectively), and $P \subset R$ and $Q \subset R$ are convex compacta; and

b) the terminal set M which can be represented in the form $M = M_0 + W_0$, where M_0 is a linear subspace of space R and W_0 is some compact convex set in space L and is the orthogonal complement of M_0 in R.

We assume that conditions $1 - 3$ defined in [1] are satisfied for problem (1.1). The notation used here conforms to that in [$1 - 3$]. We denote the sets $\pi\Phi(r) P$ and $\pi\Phi(r) Q$ by $P(r)$ and $Q(r)$, respectively.

N o t e. The formula at the bottom of p. 208 of [2] contains an error (*). It should read

$$u_i(r) \equiv u(T_1 + \varepsilon_i - r, \psi(z_i, T_1 + \varepsilon_i))$$

2. D e f i n i t i o n. We say that a total sweep [3] takes place on set $E \subset [0, +\infty)$, if for any $r \in E$ the set $w(r) = P(r) \overset{*}{-} Q(r)$ is nonempty and

$$w(r) + Q(r) = P(r) \tag{2.1}$$

(see [4, 5] for the operation over convex sets).

Let $\varphi \in \check{K}$ and $r > 0$. We denote by

$$h_P(r, \varphi) = \max_{u \in P} (\varphi \cdot \Phi(r) u) = \max_{p \in P(r)} (\varphi \cdot p) \tag{2.2}$$

$$h_Q(r, \varphi) = \max_{v \in Q} (\varphi \cdot \Phi(r) v) = \max_{q \in Q(r)} (\varphi \cdot q) \tag{2.3}$$

$$h(r, \varphi) = \max_{w \in w(r)} (\varphi \cdot w) \tag{2.4}$$

the basis spacing of sets $P(r)$, $Q(r)$ and $w(r)$, respectively, considered in L (see [5]).

*) E d i t o r ' s N o t e. In the English edition this formula appears at the top of page 196, PMM Vol. 37, № 2, 1973.

We set

$$w(r, \varphi) = \pi\Phi(r)[u(r, \varphi) - v(r, \varphi)]$$

Lemma 1. For a complete sweep of set E it is necessary and sufficient for the inequality

$$(\varphi \cdot [w(r, \varphi) - w(r, \psi)]) \geqslant 0 \qquad (2.5)$$

to be satisfied for all $r \in E$, $\varphi \in K$ and $\psi \in K$.

Proof. If a total sweep takes place on set E, then [5]

$$h(r, \varphi) + h_Q(r, \varphi) = h_P(r, \varphi), \qquad r \in E, \quad \varphi \in K \qquad (2.6)$$

We denote by $w^*(r, \varphi) \in w(r)$ the vector which yields maximum in (2.4). With the use of (2.1) and (2.6) we find that vector $w^*(r, \varphi) + \pi\Phi(r)\, v(r, \varphi)$ lies in $P(r)$ and yields the maximum in (2.2). Hence by virtue of condition 1 in [1] we have $w^*(r, \varphi) + \pi\Phi(r)\, v(r, \varphi) = \pi\Phi(r)\, u(r, \varphi)$ and, consequently, $w(r, \varphi) = w^*(r, \varphi) \in w(r)$. Similarly, $w(r, \psi) \in w(r)$. Hence (2.5) is a corollary of (2.4) and of the definition of $w^*(r, \varphi)$.

Conversely, let us assume that (2.5) is satisfied and consider the set

$$w^*(r) = \bigcup_{\varphi \in K} w(r, \varphi)$$

We denote its convex envelope by $w(r) = \operatorname{co} w^*(r)$ and the basis spacing of that envelope by $h(r, \varphi)$ (see (2.4)). We shall prove that

$$h(r, \varphi) \equiv (\varphi \cdot w(r, \varphi)) \qquad (2.7)$$

In fact, for any $w \in w(r)$ there exist $\nu + 1$ vectors [6] $\psi_1, \ldots, \psi_{\nu+1} \in K$ and numbers $\alpha_1, \ldots, \alpha_{\nu+1} \in [0, 1]$ such that

$$w = \alpha_1 w(r, \psi_1) + \ldots + \alpha_{\nu+1} w(r, \psi_{\nu+1}), \qquad 1 = \alpha_1 + \ldots + \alpha_{\nu+1}$$

Setting in formula (2.5) $\psi = \psi_i$, $i = 1, \ldots, \nu + 1$, multiplying it by α_i, and adding the derived inequalities, we obtain

$$(\varphi \cdot [w(r, \varphi) - w]) \geqslant 0 \qquad (2.8)$$

Since $w(r, \varphi) \in w(r)$, formula (2.7) is proved. Furthermore, since (2.6) follows from (2.7), Theorem 12 in [7] yields (2.1) and by virtue of Statement 2 in [4] $w(r) = P(r) \overset{*}{-} Q(r)$. This proves that a total sweep takes place on set E.

Note. In conformity with the definition of the basis spacing and Theorem 12 in [7] mentioned above, it is necessary and sufficient for vector w to lie in set $w(r)$, if inequality (2.8) is to be satisfied for all $\varphi \in K$.

3. Condition A. We say that condition A is satisfied on set $E \subset [0, +\infty)$ if for each vector $u \in P$ there exists vector $V(u) \in Q$ such that for all $r \in E$ the inclusion

$$\pi\Phi(r)[u - V(u)] \in w(r) \qquad (3.1)$$

is achieved.

Let T be an arbitrary integer. It was shown in [3] that the satisfaction of condition A along the segment $[0, T]$ is a sufficient condition of global optimal time $T(z) \leqslant T$ of the upper layer [1].

Note. If the selection of vector $V(u)$ is in accordance with the procedure in [6] so as to conform to condition A, the additional requirement for the measurability of related controls (see [3]) is automatically satisfied.

Throughout the following analysis we assume that condition 4 in [6] for $\dim P = \dim Q = \nu$. is satisfied for problem (1.1), and that the linear subspaces M_P and M_Q and vectors p and q are such that

$$P^* = P - p \subset M_P, \quad \dim M_P = \nu, \qquad Q^* = Q - q \subset M_Q, \quad \dim M_Q = \nu$$

We denote by π_P and π_Q the operators of orthogonal projections for R on M_P and M_Q, respectively. Let us consider the linear image $f(r) = \pi\Phi(r)\pi_P : M_P \to L$. Since the set $P^*(r) = f(r)P^*$ derived from the set $P(r)$ by a shift by vector $\pi\Phi(r)p$, has a nonempty interior in L (Lemma 1 in [1]), hence $f(r)$ is a mapping "onto" but, owing to $\dim M_P = \dim L$, $f(r)$ is a homeomorphism. For similar reasons the mapping $g(r) = \pi\Phi(r)\pi_Q : M_Q \to L$ is a linear homeomorphism "onto". Hence their inverse images $f^{-1}(r) : L \to M_P$ and $g^{-1}(r) : L \to M_Q$ are also linear homeomorphisms "onto" [8].

We denote by $F(r) : L \to M_P$ and $G(r) : L \to M_Q$ the linear images conjugate of $f(r)$ and $g(r)$, respectively, i.e. such that satisfy equalities

$$(F(r)x \cdot y) = (x \cdot f(r)y), \qquad (G(r)x \cdot z) = (x \cdot g(r)z)$$

for any $x \in L$, $y \in M_P$ and $z \in M_Q$. Then $F(r)$ and $G(r)$ are homeomorphisms "onto" [8], and $F(r) = \pi_P\Phi^*(r)\pi$, and $G(r) = \pi_Q\Phi^*(r)\pi$, and consequently, their inverse images $F^{-1}(r) : M_P \to L$ and $G^{-1}(r) : M_Q \to L$ are also linear homeomorphisms "onto".

If the bases are fixed in each subspace L, M_P and M_Q, the matrices of each of the above images are nondegenerate analytic matrices of order ν in the interval $(0, +\infty)$ whose elements are analytic functions of parameter r in $(0, +\infty)$. Below we assume these matrices to have been conveniently chosen, and shall deal with matrices instead of images.

We denote by $u^*(r, \varphi)$ and $v^*(r, \varphi)$ the vectors for which the expressions $(\varphi \cdot \Phi(r)u^*)$, $u^* \in P^*$ and $(\varphi \cdot \Phi(r)v^*)$, $v^* \in Q^*$ attain their respective maxima. It can be readily verified that such vectors are unique for any $r > 0$ and $\varphi \in K$, and that $u^*(r, \varphi) = u(r, \varphi) - p$ and $v^*(r, \varphi) = v(r, \varphi) - q$.

We denote by ∂ the boundary of the convex set in the plane that contains it.

N o t e. For any fixed $\theta > 0$ the image $u(\theta, \varphi) : K \to M_P + p$ is the homeomorphism of sphere K onto ∂P and $v(\theta, \varphi)$ is the homeomorphism of K ontc ∂Q. In fact, since $f(r)$ is a homeomorphism "onto", every inner point of P^* converts to an inner point of set $P^*(r) = f(r)P^*$, and every point of the boundary ∂P^* converts to a boundary point of set $P^*(r)$. Since for $\varphi \neq \psi$

$$\partial P^*(r) = \bigcup_{\varphi \in K} \pi\Phi(r)u^*(r, \varphi)$$

and $(\varphi \cdot [\pi\Phi(r)u^*(r, \varphi) - \pi\Phi(r)u^*(r, \psi)]) > 0$, (condition 1 and Lemma 1 in [1]), hence $\pi\Phi(r)u^*(r, \varphi)$ is a homeomorphism of K onto $\partial P^*(r)$ and, consequently,

$$\partial P^* = \bigcup_{\varphi \in K} f^{-1}(r)\pi\Phi(r)u^*(r, \varphi) = \bigcup_{\varphi \in K} u^*(r, \varphi) \tag{3.2}$$

and $u^*(r, \varphi)$ is a homeomorphism of K onto ∂P^*. In conformity with the definition of boundary $\partial P = \partial P^* + p$ [6], this yields

$$\partial P = \bigcup_{\varphi \in K} u(r, \varphi)$$

and $u(r, \varphi)$ is the homeomorphism of K onto ∂P. The second part of this remark is proved in a similar manner.

We denote by K_P and K_Q the unit spheres in M_P and M_Q, respectively. Formula (3.2) shows that ∂P^* is a locally convex surface. Denoting by $u^*(\psi), \psi \in K_P$ the point of surface ∂P^* at which the external normal to ∂P^* is equal to ψ, then for any $r > 0$, $\varphi \in K$ and $\psi \in K_P$

$$u^*(r, \varphi) = u^*\left(\frac{F(r)\varphi}{|F(r)\varphi|}\right), \quad u^*(\psi) = u^*\left(r, \frac{F^{-1}(r)\psi}{|F^{-1}(r)\psi|}\right) \tag{3.3}$$

Similarly, by denoting by $v^*(\psi)$, $\psi \in K_Q$ the point of surface ∂Q^* at which ψ is the external normal to ∂Q^*, we obtain

$$v^*(r, \varphi) = v^*\left(\frac{G(r)\varphi}{|G(r)\varphi|}\right), \quad v^*(\psi) = v^*\left(r, \frac{G^{-1}(r)\psi}{|G^{-1}(r)\psi|}\right) \tag{3.4}$$

Lemma 2. Condition A is satisfied in set E then and only then when for any $u \in \partial P$ there exists $V(u) \in Q$ such that for any $r \in E$ (3.1) is satisfied. To prove this lemma it is sufficient to repeat verbatim the reasoning presented in [3].

Lemma 3. If complete sweeping takes place on set E, condition A is satisfied on E, then and only then when there exists $\theta \in E$ such that for any $\varphi \in K$ and $r \in E$ the inclusion

$$\pi\Phi(r)\, y(\theta, \varphi) \in w(r) \tag{3.5}$$

is achieved. Here and in what follows $y(r, \varphi) = u(r, \varphi) - v(r, \varphi)$. In fact, for fixed $\theta \in E$ and in conformity with Note 3 the image $u(\theta, \varphi)$ is the homeomorphism of sphere K onto ∂P. Hence by virtue of Lemma 2 inclusion (3.5) ensures the satisfaction of condition A.

Conversely, if condition A is satisfied, then by virtue of (3.1)

$$\pi\Phi(\theta)\, u(\theta, \varphi) = \pi\Phi(\theta)\, V(u(\theta, \varphi)) + w, \qquad w \in w(\theta) \tag{3.6}$$

Using equalities (2.6) and (2.2) — (2.4) and the definition of vector $v(\theta, \varphi)$ [1] and the scalar multiplication of (3.6) by φ, we obtain

$$V(u(\theta, \varphi)) = v(\theta, \varphi) \tag{3.7}$$

which together with (3.1) proves (3.5).

Lemma 4. When conditions 1 — 4 are satisfied for problem (1.1) and a total sweep takes place in the half-interval $(0, T]$, then to have condition A satisfied in $(0, T]$ it is necessary and sufficient if condition A is satisfied on every subset $I \subset (0, T]$, which has in $(0, T]$ a limit point. The necessity of this condition is evident. Let us prove that it is sufficient.

For fixed $\tau \in I$ and any $r \in I$, φ, and $\psi \in K$ we have (see (2.8) and (3.5))

$$(\psi \cdot \Phi(r)\,([u(r, \psi) - v(r, \psi)] - [u(\tau, \varphi) - v(\tau, \varphi)])) \geqslant 0 \tag{3.8}$$

Setting $\varphi_r = F^{-1}(r)\, F(\tau)\varphi$ and $\psi = \varphi_r / |\varphi_r|$, in accordance with (3.3) and (3.4) we obtain

$$u(r, \psi) = u^*(F(r)\psi / |F(r)\psi|) + p = u^*(F(\tau)\varphi / |F(\tau)\varphi|) + p =$$
$$u(\tau, \varphi)\quad v(\tau, \varphi) = v(r, \psi_r), \quad \psi_r = h(r)\psi / |h(r)\psi|$$

where

$$h(r) = G^{-1}(r)\, G(\tau)\, F^{-1}(\tau)\, F(r) : L \to L \tag{3.9}$$

is a nondegenerate linear transformation of space L.

Substituting the obtained expressions into formula (3.8) and using the local convexity of surface $\pi\Phi(r)\, v(r, K)$ (Lemma 1 in [1]), we obtain $0 \geqslant (\psi \cdot \Phi(r)\,[v(r, \psi) - v(r, \psi_r)]) \geqslant c_2(\psi \cdot [\psi - \psi_r]) \geqslant 0$. This shows that $\psi_r \equiv \psi$ and, consequently, for any $\psi \in K$ (since φ is arbitrary)

$$h(r)\psi = |h(r)\psi|\psi$$

It follows from this that $|h(r)\psi| \equiv \lambda(r)$ is, as in [2], independent of ψ.

As pointed out above, $h(r)$ is a matrix whose elements are analytic functions of parameter $r \in (0, +\infty)$, $h(r) \equiv \lambda(r) E$ and $r \in I$, i.e. all elements that do not lie on the principal diagonal of matrix $h(r)$ vanish at all points of set I which shows that they are identically zeros. The diagonal elements of matrix $h(r)$ are the same on set I, hence they are the same for all r. Thus

$$h(r) \equiv \lambda(r) E, \quad r \in (0, +\infty) \tag{3.10}$$

Hence it follows from formula (3.9) that the equality $G(\tau) F^{-1}(\tau) = \lambda(r) G(r) F^{-1}(r)$ holds for all $r \in (0, +\infty)$. Passing to conjugate transformations from this we obtain

$$B = f^{-1}(\tau) g(\tau) \equiv \lambda(r) f^{-1}(r) g(r)$$

Since $h(r)$ is a matrix which is nondegenerate for any $r \in (0, +\infty)$ and $\lambda(\tau) = 1$, hence $\lambda(r) > 0$ for $0 < r < +\infty$. Setting $\alpha(r) = 1/\lambda(r)$, we finally obtain

$$g(r) = \alpha(r) f(r) B, \quad 0 < r < +\infty \tag{3.11}$$

with $B : M_Q \to M_P$ an nondegenerate matrix.

If r is any arbitrary number in the interval $(0, T]$, then, setting

$$\theta = \tau, \quad w = \pi\Phi(r) y(\theta, \varphi) \text{ and } \varphi_r = F^{-1}(r) F(\tau) \varphi,$$

and using (3.3), (3.4) and (3.10) and Lemma 1, we obtain for any $\psi^* \in K$ the inequality

$$(\psi^* \cdot [w(r, \psi^*) - w]) = (\psi^* \cdot [w(r, \psi^*) - w(r, \varphi_r / |\varphi_r|)]) \geqslant 0$$

which by virtue of Note 2 yields (3.5). Hence, in conformity with Lemma 3 condition A is satisfied in $(0, T]$. The lemma is proved.

Lemma 5. If a complete sweep takes place in $(0, T]$ and condition A is not satisfied, then for any $\theta \in [0, T]$ there exists a subset $K(\theta) \subset K$, which consists of a finite number of cross sections of sphere K by subspaces of dimension $\leqslant \nu - 1$ (the emptiness of set $K(\theta)$ is not excluded) such that for any $\varphi \in K \setminus K(\theta)$ there exists a subset $E(\varphi, \theta) \subset (0, T]$, which has no limit points in $(0, T]$ such that

$$\pi\Phi(r) y(\theta, \varphi) \notin w(r) \tag{3.12}$$

for all $r \in [0, T] \setminus E(\varphi, \theta)$.

Proof. Let $r > 0$ and $\varphi \in K$. Inclusion (3.5) is achieved then and only then when for any $\psi \in K$ the inequality (3.8) is valid ($\tau = \theta$) is assumed). From this, as from the proof of Lemma 4, we have $G^{-1}(r) G(\theta) \varphi = \gamma(r, \varphi) F^{-1}(r) F(\theta) \varphi$, where $\gamma(r, \varphi) > 0$ or what is the same

$$H(r) \varphi = \gamma(r, \varphi) \varphi \tag{3.13}$$

where $H(r) = F^{-1}(\tau) F(r) G^{-1}(r) G(\tau) : L \to L$ is the nondegenerate linear transformation of space L.

We assume that $\varphi \in N(\theta)$, then and only then when equality (3.13) is satisfied with respect to r on some subset $D(\varphi)$ of the interval $(0, T)$ which has in $(0, T]$ a limit point. If the reference in L is chosen so that its first basis vector is vector φ, then only the first element of the first column of matrix $H(r)$ is nonzero for all $r \in D(\varphi)$. The elements of matrix $H(r)$ are analytic functions of parameter r, hence all elements of the first column of matrix $H(r)$, except the first, are identically zeros. This shows that equality (3.13) is satisfied for all $r \in (0, +\infty)$.

If among the vectors from $N(\theta)$ there are no ν linearly independent vectors, the set $N(\theta)$ lies in some $(\nu-1)$-dimensional subspace $L(\theta)$ of space L. We then set $K(\theta) = L(\theta) \cap K$. If, however, $\varphi_1, \ldots, \varphi_\nu \in N(\theta)$ form the basis in L, then matrix $H(r)$ in that basis is for all $r > 0$ of the diagonal form

$$H(r) = \operatorname{diag}(\lambda_1(r), \ldots, \lambda_\nu(r)) \tag{3.14}$$

where $\lambda_i(r) = \gamma(r, \varphi_i)$ are analytic functions of parameter r.

Let $i_1 < \ldots < i_p$ be an arbitrary subset of set $\{1, 2, \ldots, \nu\}$. We denote by $L(i_1, \ldots, i_p)$ the linear envelope of vectors $\varphi_{i_1}, \ldots, \varphi_{i_p}$ and set

$$K(\theta) = \bigcup_{p<\nu} (L(i_1, \ldots, i_p) \cap K)$$

Let $\varphi \notin K(\theta)$. We shall show that there exists a subset $F(\varphi, \theta) \subset (0, T]$ which has no limit points in $(0, T]$, and such that (3.12) is satisfied for any $r \in (0, T] \setminus E(\varphi, \theta)$. We prove this by contradiction. Let us assume that $D(\varphi) \subset (0, T]$ has a limit point in $(0, T]$ and is such that (3.5) is satisfied for all $r \in D(\varphi)$, then equality (3.13) is satisfied in $D(\varphi)$ and, consequently, $\varphi \in N(\theta)$. Hence by distributing vector φ over the basis $\varphi_1, \ldots, \varphi_\nu \in K(\theta)$, we obtain

$$\varphi = \alpha^1\varphi_1 + \ldots + \alpha^\nu\varphi_\nu$$

$$\sum_{i=1}^{\nu} [\alpha^i\gamma(r, \varphi)]\varphi_i = \gamma(r, \varphi)\varphi = H(r)\varphi = \sum_{i=1}^{\nu} [\alpha^i\lambda_i(r)]\varphi_i, \quad 0 < r < +\infty$$

which implies that for each $i = 1, \ldots, \nu$ either $\alpha^i = 0$, or $\lambda_i(r) \equiv \gamma(r, \varphi)$.

Let us show that there exists an i_0 such that $\alpha^{i_0} = 0$. This implies that $\varphi \in L(1, \ldots, i_0 - 1, i_0 + 1, \ldots, \nu) \cap K \subset K(\theta)$, which is a contradiction that proves the lemma.

In fact, if $\alpha^i \neq 0$, $i = 1, \ldots, \nu$, all functions $\lambda_i(r)$ are the same, i.e. $\lambda_i(r) \equiv \lambda(r)$, $i = 1, \ldots, \nu$, $0 < r < +\infty$.

Hence by virtue of (3.14) $(1/\lambda(r))H(r) = H_1 \cdot H_2 = E$, where

$$H_1 = (1/\lambda(r))F^{-1}(\tau)F(r) \text{ and } H_2 = G^{-1}(r)G(\tau).$$

This means that $H_2 = H_1^{-1}$ and, consequently, $(1/\lambda(r))h(r) = H_2 \cdot H_1 = E$.

We have obtained formula (3.10) which ensures that condition A is satisfied in $(0, T]$ (see the proof of Lemma 4).

4. Let us assume that the condition of complete sweep is not satisfied in the half-open interval $(0, T]$ or, if the complete sweep takes place, condition A is not satisfied.

For any $\theta \in (0, T]$, $r \in (0, T]$ and $\varphi \in K$ we denote by $O(r, \theta, \varphi)$ the set of all vectors $\psi \in K$ for which

$$\mu(r, \psi, \theta, \varphi) \equiv (\psi \cdot \pi\Phi(r)[y(r, \psi) - y(\theta, \varphi)]) < 0 \tag{4.1}$$

Note that for certain sets of r, θ, φ the set $O(r, \theta, \varphi)$ is empty. Lemmas 1 and 5 ensure the nonemptiness of set $O(r, \theta, \varphi)$ for at least one set of r, θ, φ.

We assume that for problem (1.1) the following conditions are satisfied.

C o n d i t i o n B. There exists $\theta \in (0, T)$, $r \in (0, T)$, $r \neq 0$, $\varphi \in K$, $\psi \in O(r, \theta, \varphi)$ and $z_0 \in R$ such that

$$\lambda(z_0, t) < 0, \quad t \in \begin{cases} [0, \theta) \cup (\theta, r), & \theta < r \\ [0, r) \cup (r, \theta), & r < \theta \end{cases} \tag{4.2}$$

$$\pi\Phi(\theta)z_0 = W(\theta, \varphi), \quad \pi\Phi(r)z_0 = W(r, \psi)$$

Theorem 1. If condition B is satisfied for problem (1.1), there exists in R a point z_* at which the time $T(z_*) \leqslant T$ is nonoptimal.

Proof. We set $\alpha = \Phi^*(\theta)\varphi$, $\beta = \Phi^*(r)\psi$ and $b = |\alpha|\beta - |\beta|\alpha$, and shall show that

$$b \neq \bar{0} \tag{4.3}$$

by contradiction. If $b = \bar{0}$, then $\beta = c\alpha$, where $c = |\beta| / |\alpha|$ and, consequently, $(\beta \cdot u) \equiv c(\alpha \cdot u)$ for any $u \in P$. Hence the maxima

$$\max_{u \in P} (\varphi \cdot \Phi(\theta) u), \quad \max_{u \in P} (\psi \cdot \Phi(r) u) \tag{4.4}$$

are reached on one and the same vector $u_0 \in P$ which is unique by virtue of condition 1 in [1]. From this we have $u(\theta, \varphi) = u_0 = u(r, \psi)$. For the same reason $v(\theta, \varphi) = v(r, \psi)$ and, consequently, the left-hand side of inequality (4.1) vanishes, which is a contradiction.

It follows from formulas (4.1) — (4.3) that z_0 is a "singular" point when $\theta < r$ hence by Theorem 2 in [2] there exists in R a point z_* at which the time $T(z_*)$ is nonoptimal. Furthermore, the construction itself of the proof of that theorem yields the inequality $T(z_*) < r < T$. Thus the theorem is proved for the case of $\theta < r$.

The case of $r < \theta$. Let us prove a more general statement.

Lemma 7. Let $z_0 \in R$, k be a natural number, and $0 < r_1 < \ldots < r_k < \theta < T$ be such that

$$\pi\Phi(r_i) z_0 = W(r_i, \psi_i), \quad i = 1, \ldots, k, \quad \pi\Phi(\theta) z_0 = W(\theta, \varphi) \tag{4.5}$$

$$\lambda(z_0, t) < 0, \quad t \in [0, r_1) \cup \ldots \cup (r_i, r_{i+1}) \ldots \cup (r_k, \theta) \tag{4.6}$$

$$\mu_i = \mu(r_i, \psi_i, \theta, \varphi) < 0, \; i = 1, \ldots, k \tag{4.7}$$

Then there exists in R a point z_* at which the time $T(z_*) \leqslant T$ is nonoptimal.

Proof. Let $\varepsilon_0 > 0$ be so small that for $0 < \varepsilon < \varepsilon_0$ we have $\tau = r_1 - \varepsilon > 0$ and $T^\varepsilon = \theta + \varepsilon < T$. We set

$$z_\varepsilon = \Phi(-\varepsilon)\left[z_0 + \int_0^\varepsilon \Phi(s) y(\theta + s, \varphi)\, ds\right]$$

Then

$$\lambda(z_\varepsilon, t) \leqslant \Big(\varphi(t, \varepsilon) \cdot \Big\{W(t, \varphi(t, \varepsilon)) - \pi\Phi(t - \varepsilon) z_0 -$$
$$\int_0^\varepsilon \pi\Phi(t - \varepsilon + s) y(\theta + s, \varphi)\, ds\Big\}\Big) = \lambda(z_0, t - \varepsilon) - g(t, \varepsilon), \quad \varepsilon \leqslant t \leqslant T$$

where $\varphi(t, \varepsilon) = \psi(z_0, t - \varepsilon)$

$$g(t, \varepsilon) = \int_{t-\varepsilon}^{t} (\varphi(t, \varepsilon) \cdot \pi\Phi(s)[y(\theta + s - t + \varepsilon, \varphi) - y(s, \varphi(t, \varepsilon))])\, ds$$

Since $\varphi(t, \varepsilon) \to \psi_i$ when $t \to r_i$ and $\varepsilon \to 0$, and since $|s - t + \varepsilon| \leqslant \varepsilon \to 0$ uniformly with respect to $s \in [t - \varepsilon, t]$ and $t \in [\varepsilon, T]$ when $\varepsilon \to 0$, hence owing to the continuity of all functions in the left-hand side of inequality (4.7), there exists a δ, $0 < \delta < \min\{\varepsilon_0, r_1 / 3, \; (\theta - r_k) / 5, \min\limits_{1 \leqslant i < k} (r_{i+1} - r_i) / 5\}$, such that for all

$$t \in D_0 = \bigcup_{i=1}^{k} [r_i - 2\delta, r_i + 2\delta]$$

and all $\varepsilon \in (0, \delta)$ the inequality $g(t, \varepsilon) < 0$ is satisfied. This together with (4.6) yields

$$\lambda(z_\varepsilon, t) < 0, \quad t \in D_0, \; \varepsilon \in (0, \delta) \tag{4.8}$$

Let $T_\varepsilon = T(z_\varepsilon)$. We shall show that

$$T_\varepsilon \to \theta, \quad \varepsilon \to +0 \tag{4.9}$$

First, we note that $\pi\Phi(\theta+\varepsilon)z_\varepsilon = W(\theta+\varepsilon, \varphi)$, so that $T_\varepsilon \leqslant \theta+\varepsilon$ (see [1]). Hence, if (4.9) does not apply, there exist a number γ, $0 \leqslant \gamma < \theta$, and a sequence $\varepsilon_i \to +0$, such that $\lim_{i\to\infty} T_i = \gamma$ and

$$\lambda(z_i, T_i) = 0, \qquad i = 1, 2, \ldots \tag{4.10}$$

where $z_i = z_{\varepsilon_i}$ and $T_i = T(z_i)$.

Since $z_i \to z_0$, hence, owing to the continuity of function $\lambda(z, t)$, $\lambda(z_0, \gamma) = 0$, and there exists a natural number m $(1 \leqslant m \leqslant k)$ such that $\gamma = r_m$ and, consequently, the inclusions $\varepsilon_i \in (0, \delta)$ and $T_i \in D_0$ are achieved for all reasonable great i. However, this means that (4.10) contradicts (4.8).

Let us prove that for fairly small ε the necessary condition of optimum (Theorem 2 in [1]) is not satisfied at point z_ε, and this will conclude the proof of the theorem and Lemma 7. We prove it by contradiction. Let for any $\varepsilon > 0$

$$I(2\varepsilon, \tau) = \lambda(x_\varepsilon, \tau) \geqslant 0 \tag{4.11}$$

$$x_\varepsilon = \Phi(2\varepsilon)\left[z_\varepsilon - \int_0^{2\varepsilon} \Phi(-s)\, y(T_\varepsilon - s, \varphi_\varepsilon)\, ds\right], \quad \varphi_\varepsilon = \varphi(z_\varepsilon)$$

Setting $\psi_\varepsilon = \psi(x_\varepsilon, \tau)$, after some transformations (using the notation $r = r_1$ and $\psi = \psi_1$), we obtain

$$0 \geqslant I(2\varepsilon, \tau) = a_1 + a_2 + a_3 + a_4 \geqslant a_2 + a_3 + a_4 \tag{4.12}$$

where $a_1 = (\psi_\varepsilon \cdot [W(\tau, \psi_\varepsilon) - W(\tau, \psi)]) \geqslant 0$ (Lemma 2 in [1]) and

$$a_2 = -\left(\psi_\varepsilon \cdot \int_{r-\varepsilon}^{r} \Phi(s)\, y(s, \psi)\, ds\right), \quad a_3 = -\left(\psi_\varepsilon \cdot \int_0^{\varepsilon} \Phi(r+\varepsilon-s)\, y(T^\varepsilon - s, \varphi)\, ds\right)$$

$$a_4 = \left(\psi_\varepsilon \cdot \int_0^{2\varepsilon} \Phi(r+\varepsilon-s)\, y(T_\varepsilon - s, \varphi_\varepsilon)\, ds\right)$$

Dividing inequality (4.12) termwise by $\varepsilon > 0$, passing to limit $\varepsilon \to 0$, and using relationships $\psi_\varepsilon \to \psi$, $T^\varepsilon \to \theta$, and (4.9) with its corollaries $\varphi_\varepsilon \to \varphi$, when $\varepsilon \to 0$ and, also, the uniform continuity of functions $u(r, \varphi)$ and $v(r, \varphi)$ in $[\delta, T] \times K$ (see [2]), we obtain

$$(\psi \cdot \Phi(r)[y(\theta, \varphi) - y(r, \psi)]) \leqslant 0 \tag{4.13}$$

which contradicts (4.5).

Note. It follows from the proof of the theorem that for the time $T(z)$ of the upper layer to be globally optimal it is necessary that the inequality (4.13) is satisfied at any point $z_0 \in R$ that satisfies condition (4.2).

5 Let us assume that in addition to conditions 1 – 4 the following condition (see [2], Sect. 5) is satisfied for problem (1.1):

$$\pi\Phi(r)\, u = \pi\Phi(r) p + f(r)\, Au^*, \quad u^* = u - p, \quad u \in P$$
$$\pi\Phi(r)\, v = \pi\Phi(r)\, q + g(r)\, Bv^*, \quad v^* = v - q, \quad v \in Q$$

where $A : M_P \to L$ and $B : M_Q \to L$ are linear homeomorphisms "onto", $f(r)$ and $g(r)$ are analytic functions that are positive in some half-open interval $(0, T]$, vectors p

and q are defined in Sect. 3, $AP^* = BQ^* = S$, $W_0 = lS$ with $l \geqslant 0$ and $l > 0$, when $f(r) < g(r)$ in some right-hand half-neighborhood of zero.

In that case ([2])

$$u^*(r, \varphi) \equiv U(\varphi), \quad v^*(r, \varphi) \equiv V(\varphi), \quad AU(\varphi) = \varphi$$
$$BV(\varphi) = \varphi, \quad W(t, \varphi) = h(t)\varphi + \Psi(t)$$

where

$$h(t) = l + \int_0^t (f(r) - g(r))\, dr, \quad \Psi(t) = \pi \int_0^t \Phi(r)(p - q)\, dr$$

We assume that $h(t) > 0$ and $t \in (0, T)$.

Theorem 2. Let problem (1.1) under conditions of this Section be such that one of the two sets of three $\{f(r), g(r), f'(r)\}$ or $\{f(r), g(r), g'(r)\}$ is linearly independent in $[0, T]$. We then have the following alternatives: either condition A is satisfied in $(0, T]$, i.e.

$$f(r) \geqslant g(r), \quad 0 \leqslant r \leqslant T \tag{5.1}$$

and then the time $T(z) \leqslant T$ is optimum [2, 9, 10], or (5.1) is not satisfied, i.e. condition A does not apply and there exists in R a point z_* at which the time $T(z_*) \leqslant T$ is not optimal.

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